THE CONSTRUCTIVIST RETROANALYST

Alexander George

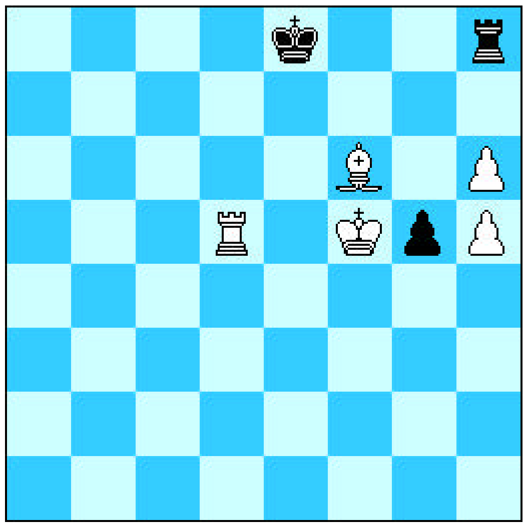
Constructivism in logic or mathematics issues from the perspective that refuses to allow a general appeal to the determinate nature of mathematical reality in the course of proving assertions. Rather, a particular proof, or construction, must accompany each claim, the precise nature of which will vary with the logical structure of the claim being established.

For instance, in order to prove a disjunction "*X* or *Y*," the constructivist demands nothing less than a proof of *X* or a proof of *Y*. The non-constructive logician – sometimes called the "classical" logician – makes no such demand; he might be willing to accept an assertion of that form even though he is neither in a position to prove *X* nor prepared to prove *Y*.

Consider the particular disjunction "*X* or not-*X*." A classical logician will accept every such disjunction, whatever *X* might be; indeed, this is often known as the *Law of the Excluded Middle*. This is because such a logician imagines that, for any meaningful statement *X*, the mathematical universe is such that either *X* is true or *X* is false. The constructivist, by contrast, makes no such assumption: if one wishes to assert "*X* or not-*X*" then one must either present a proof of *X* or present a proof of not-*X*. It might well be that one is neither in a position to prove *X* nor in a position to prove not-*X*; for the constructivist, there would then be nothing for it but to refrain from asserting the disjunction "*X* or not-*X*."

These distinctions can be brought out nicely by considering an example from retrograde analysis, a particular kind of chess problem in which the solver must use her powers of deduction, and of course her knowledge of the rules of chess, to determine facts about the history of the position with which she is presented. Consider the following, for example:

W. Langstaff, *The Chess Amateur*, 1922



White to mate in 2

The classical problemist might reason as follows: Either the history of the game is such that Black can castle (viz., neither its king nor its rook has moved) or it is such that Black cannot castle. If Black can castle, then Black's last move must have been Pg7-g5[1] (for Black has moved neither king nor rook, and Pg6-g5 would have placed White in an impossible checked position). But in that case, White can move 1.Ph5xPg5 *en passant* (*e.p.*), leading to mate the following move (either 2.Rd8 or, if Black castles, Ph7). On the other hand, if Black cannot castle, then White can move 1.Ke6 (but not 1.Ph5xPg5 *e.p.* since we cannot now show that

[1] Conventionally, the columns are labeled "a" to "h", moving from left to right. And the rows are labeled "1" to 8", moving from bottom to top; thus, in the above position the Black king is standing on e8 and the White King on f5.

Black’s last move must have been Pg7-g5), which leads to mate (namely, 2.Rd8). Summarizing: either White can capture *e.p.* or Black cannot castle; if the first, then 1.Ph5xPg5*e.p.* mates; if the second, then 1.Ke6 mates; therefore, either 1.Ph5xPg5*e.p.* mates or 1.Ke6 mates – even though there is no way for us to know which of these two moves is the actual keymove.

The “constructivist problemist” is willing to agree that *if* White can *e.p.* capture *then* 1.Ph5xPg5*e.p.* mates, and also that *if* Black cannot castle *then* 1.Ke6 mates. To this degree, he can sign on to the “classical” analysis. But he cannot go the extra step and conclude, as his classical counterpart does, that either 1.Ph5xPg5*e.p.* mates or 1.Ke6 mates. This is because the constructive retroanalyst will not assert the disjunction “White can capture *e.p.* or Black cannot castle.” He will not assert this because he cannot establish either of the two disjuncts: he cannot show that White can capture *e.p.*, nor can he show that Black cannot castle.

One might wish to object to the constructivist that surely we *can* establish that in Langstaff’s problem it cannot both be the case that White cannot capture *e.p.* and Black can castle. And the constructivist will agree: the assumption that Black can castle forces the conclusion that Black’s last move was a double pawn move which permits White to capture *en passant*. Thus the assumption that both White cannot capture *e.p.* and Black can castle leads to a contradiction; and that indeed justifies our claim that it cannot be that both conditions hold. But this claim – which is of the form “not-(not-*X* and *Y*),” where *X* is “White can capture *e.p.*” and *Y* is “Black can castle” – does <u>not</u> force the constructive retroanalyst to agree that either White can capture *e.p.* or Black cannot castle – which is a claim of the different form “*X* or not-*Y*.” For this inference – from “¬(ϕ   ψ)” to “¬ϕ ¬ψ” – is precisely one whose general validity is disputed by the constructivist.[2]

[2] The particular reasoning here also involves moving from “not-not-ϕ” to “ϕ,” which is yet another inference disputed by the constructivist.

Likewise, one might try to urge upon the constructivist retroanalyst that surely *if* Black can castle *then* White can capture *en passant*. And furthermore, “if *Y* then *X*” is logically equivalent to “not-*Y* or *X*”! Does this not force the constructivist to accept the disjunction? No again. It is true that the constructivist will agree with the conditional claim: for any proof that Black can castle can easily be transformed into a proof that White can capture *en passant*. But the sticking point is that for the constructivist “$\phi \rightarrow \psi$” does <u>not</u> logically entail “$\neg\phi \quad \psi$” : for an ability to transform any proof of ϕ into a proof of ψ might not give one the means either to prove $\neg\phi$ or to prove ψ.[3]

Thus, while the constructivist retroanalyst can agree to many claims his classical counterpart makes with regard to the logic of Langstaff’s problem, he cannot take the final step and state the solution disjunctively; he cannot baldly assert “either 1.Ph5xPg5*e.p.* mates or 1.Ke6 mates.”[4]

DEPARTMENT OF PHILOSOPHY
AMHERST COLLEGE
AMHERST, MA 01002

---

[3] For more information about constructivism – its philosophical basis and the logic and mathematics attending it -- the reader may wish to consult Alexander George and Daniel J. Velleman, *Philosophies of Mathematics*, Blackwell, 2002; see especially Chapter 4, “Intuitionism.”

[4] I do not believe that the idea of using retroanalysis problems to illustrate the difference between classical and constructive reasoning is original to me, but all my efforts to locate a source have failed.